ESTIMATES OF $\| e^{itf} \|_{A(\Gamma)}$, WHEN $\Gamma \subset \mathbb{R}^n$ IS A CURVE AND f IS A REAL-VALUED FUNCTION

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ABSTRACT

Γ is a simple curve in $\mathbb{R}^n$ with an equation $x = \gamma(u)$, $u \in [0.1]$, $\gamma \in C^k$, where for some k, $2 \leqq k - 1 \leqq n$, $\gamma'(u), \gamma''(u), \ldots, \gamma^{k-1}(u)$ are linearly independent for every u. It is then proved that if $t \in \mathbb{R}$ and f is a real-valued function in $C^k(\Gamma)$, $t^{-1/k} \|e^{itf}\|_{A(\Gamma)}$ is bounded as $|t| \to \infty$. An example shows that the estimate cannot be improved in general, when $n = 2, k = 3$. The result is interpreted in terms of properties of the space of pseudomeasures on Γ.

For every complex-valued function g on the closed set $E \subset \mathbb{R}^n$, A_g is the set of functions $h \in A(\mathbb{R}^n) = \mathscr{F}L^1(\mathbb{R}^n)$, which coincide with g when restricted to E. $A(E)$ is the space of all g with non-empty A_g, and for $g \in A(E)$, we put

$$\| g \|_{A(E)} = \inf_{h \in A_g} \| h \|_{A(\mathbb{R}^n)}.$$

In an analogous way we introduce $B(E)$ and $\| g \|_{B(E)}$, using instead af $A(\mathbb{R}^n)$ the larger space $B(\mathbb{R}^n) = \mathscr{F}M(\mathbb{R}^n)$. $g \in A(E)$ implies that $g \in B(E)$ and $\| g \|_{A(E)} = \| g \|_{B(E)}$ (cf. Shapiro [5], p. 121).

Let Γ be a compact interval in $\mathbb{R}$ and f a real-valued function in $C^1(\Gamma)$. It is then a well-known and easy consequence of an inequality of Carlson [2] that $e^{itf} \in A(\Gamma)$ for every $t \in \mathbb{R}$, and that for some constant K

$$\| e^{itf} \|_{A(\Gamma)} \leqq K(1 + |t|)^{\frac{1}{2}}, \qquad t \in \mathbb{R}. \tag{1}$$

If $f \in C^2(\Gamma)$ is not a linear function on Γ, the estimate (1) is sharp in the sense that

$$\| e^{itf} \|_{A(\Gamma)} \geqq K'(1 + |t|)^{\frac{1}{2}}, \qquad t \in \mathbb{R}, \tag{2}$$

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for some positive constant K' (Leibenson [4]).

Our aim is to find corresponding estimates in the case when Γ is a curve in $\mathbb{R}^n$ $n \geqq 2$. Applying the one-dimensional result, it is easy to prove that (1) holds as well if $f \in C^1(\Gamma)$ and Γ is a simple C^1 curve of finite length. The case when Γ is a line segment is of course equivalent to the one-dimensional case, hence (1) can not be improved in general. The following theorem shows however, that we can lower the exponent in (1), if we assume higher differentiability of f and Γ and if we impose conditions which prevent Γ from being too degenerated.

THEOREM 1. *Let $2 \leqq k - 1 \leqq n$ and let $\Gamma \subset \mathbb{R}^n$ be a simple curve with equation $x = \gamma(u)$, $u \in [0,1]$, $\gamma \in C^k$, where $\gamma'(u), \gamma''(u), \cdots, \gamma^{(k-1)}(u)$ are linearly independent for every $u \in [0,1]$. Then there exists, for every real-valued $f \in C^k(\Gamma)$, a constant K such that*

$$\| e^{itf} \|_{A(\Gamma} \leqq K(1 + |t|)^{1/k}, \qquad t \in \mathbb{R}. \tag{3}$$

PROOF. Using a partitioning of the unit we find by a standard procedure that it is sufficient if we can, for every $x \in \Gamma$, prove the theorem with Γ replaced by a suitably chosen closed neighborhood of x with respect to Γ, and with f restricted to that neighborhood. Hence, after a change of parameter and coordinates, we can conclude that it suffices to prove the theorem in the case when $\gamma_1(u) = u$ for every $u \in [0,1]$ and when the determinant $|\gamma_i^{(j)}(u)|$, $1 \leqq i \leqq k-1$, $1 \leqq j \leqq k-1$, does not vanish on $[0,1]$. Here γ_i denotes for every i, $1 \leqq i \leqq n$, the i-th component of γ.

Assuming this we extend γ to a slightly larger interval $[-\varepsilon, 1+\varepsilon]$, $\varepsilon > 0$, so that all properties mentioned hold in this interval as well. This is of course possible, and it suffices to prove (2) in the case when $m = [t^{1/k}] \geqq 2/\varepsilon$. A continuous function ϕ_m on $\mathbb{R}$ is defined by the conditions

$$\phi_m = \begin{cases} 1 \text{ on } [0, 1/2m] \\ 0 \text{ on }]-\infty, -1/2m] \cup [1/m, \infty[\\ \text{linear on each of the two remaining intervals.} \end{cases}$$

For every p, $0 \leqq p \leqq m$, $\phi_{m,p}$ denotes the function on $\mathbb{R}$ with values

$$\phi_{m,p}(x_1) = \phi_m(x_1 - p/m), \qquad x_1 \in \mathbb{R}.$$

Obviously,

$$\sum_{p=0}^{m} \phi_{m,p}(x_1) = 1, \tag{4}$$

when $x_1 \in [0,1]$.

ψ_p denotes for $0 \leqq p \leqq m$ the function or $\mathbb{R}$ with values

$$\psi_p(x_1) = \phi_{m,p}(x_1) \exp it\{f \circ \gamma(x_1) - \sum_{l=1}^{k-1} a_{p,l}\gamma_l(x_1)\}, \qquad x_1 \in \mathbb{R},$$

where $a_{p,l}$ are real constants to be chosen later, and where the value is defined as 0 whenever the first factor vanishes even if the second factor is not defined. Let us now form the function g on $\mathbb{R}^n$ with values

$$g(x) = \sum_{p=0}^{m} \psi_p(x_1) \exp it(\sum_{l=1}^{k-1} a_{p,l}x_l), \qquad x \in \mathbb{R}^n.$$

Due to (4), the restriction of g to Γ is e^{itf}. Hence

$$\| e^{itf} \|_{A(\Gamma)} = \| e^{itf} \|_{B(\Gamma)} \leqq \| g \|_{B(\mathbb{R}^n)} \leqq \sum_{p=0}^{m} \| \psi_p \|_{A(\mathbb{R})}$$

by well-known properties of Fourier-Stieltjes transforms. Hence (3) is proved if we can show that, for every m and p, the numbers $a_{p,l}$ can be chosen so that $\| \psi_p \|_{A(\mathbb{R})}$ has a bound, uniform in m and p.

Carlson's inequality, valid for a class of complex-valued functions ψ on $\mathbb{R}$ containing our functions ψ_p has the form

$$\| \psi \|^2_{A(\mathbb{R})} \leqq C \| \psi \|_{L^2} \| \psi' \|_{L^2} ,$$

where C is a constant. Hence we obtain

$$\| \psi_p \|^2_{A(\mathbb{R})} \leqq C \cdot \frac{3}{2m} \cdot \sup | \psi_p | \cdot \sup \psi'_p |$$

$$\leqq C \cdot \frac{3}{2m} (\sup | \phi'_{m,p} | + \sup_{[(2p-1)/m,(p+1/m]} | t | \, | (f \circ \gamma)' - \sum_{l=1}^{k-1} a_{p,l}\gamma'_l |).$$

By the assumption on the determinant $|\gamma_l^{(j)}|$, we can choose $a_{p,l}$ so that the function $f \circ \gamma - \sum_{l=1}^{k-1} a_{p,l}\gamma_l$ has vanishing derivatives of orders $1,2,\cdots,k-1$ for $x_1 = p/m$. The values of $a_{p,l}$ are then uniformly bounded in m and p, and hence Taylor's formula shows that the second term within the parenthesis is bounded by $C | t | m^{-(k-1)} \leqq C_1 m$, for constants C and C_1, independent of m and p. Since $| \phi'_{m,p} | \leqq 2m$, $\| \psi_p \|_{A(\mathbb{R})}$ is bounded and the theorem is proved.

The following theorem gives an example of a case where Theorem 1 is the best possible.

THEOREM 2. *If $\Gamma \subset \mathbb{R}^2$ is an arc of a circle and f is real-valued and linear with respect to the arc length, there exists a positive constant K', such that*

$$\| e^{itf} \|_{A(\Gamma)} \geqq K'(1 + |t|)^{\frac{1}{3}}, \qquad t \in \mathbb{R}.$$

PROOF. We shall prove that if S^1 is the unit circle in the z-plane and if $g(z) = g(re^{i\theta}) = e^{i\theta}$, when $r = 1$, then

$$\| g^m \|_{A(S^1)} = (\sup_{x \in \mathbb{R}} | J_m(x) |)^{-1}, \tag{5}$$

for every $m \in \mathbb{Z}$. Now, the value of the right hand member is $\geqq Cm^{\frac{1}{3}}$, for some positive constant C and $m \neq 0$. It is difficult to find a convenient reference in the huge literature on the asymptotic behavior of Bessel functions, but the fact is easily established by estimating

$$2\pi J_m(x) = \int_{-\pi}^{\pi} e^{i(x\sin\phi - m\phi)} d\phi, \qquad x \in \mathbb{R},$$

by elementary methods. The theorem follows then from (5) by a partitioning of the unit.

To prove (5) we first observe that there exists a $h_m \in B(\mathbb{R}^2)$ such that its restriction to S^1 is g^m, and such that

$$\| h_m \|_{B(\mathbb{R}^2)} = \| g^m \|_{B(S^1)} = \| g^m \|_{A(S^1)}.$$

The main idea behind the proof of this stems from Beurling [1]: for a full discussion of the details, see Shapiro [5], p. 120–121. We call such a function h_m a minimal extrapolation of g^m. Denoting by T_ϕ, $\phi \in \mathbb{T}$, the transformation

$$(T_\phi g)(z) = e^{im\theta}\, g(ze^{-i\theta}),$$

defined for all complex-valued functions g on $\mathbb{R}^2 = \mathbb{Z}$, we find that $T_\phi h_m$ is a minimal extrapolation of g^m, and the same must then hold for the function k_m, defined by

$$k_m(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_\phi g(re^{i\theta}) d\phi.$$

Then $T_\phi k_m = k_m$, for every $\phi \in \mathbb{T}$. But this implies that the measure μ_m with the Fourier-Stieltjes transform k_m has to be of the form

$$d\mu_m = e^{im\psi} d\psi dv(r), \tag{6}$$

where (r,ψ) are polar coordinates in the dual $\mathbb{R}^2$. Since k_m is a minimal extrapolation of g^m we have

$$\| g^m \|_{A(S^1)} = \| k_m \|_{B(\mathbb{R}^2)} = 2\pi \int_0^\infty | dv(r) |, \tag{7}$$

and

$$1 = k_m(1) = \int_0^\infty \left(\int_{-\pi}^{\pi} e^{im\psi - ir\cos\psi} d\psi \right) dv(r) \leqq \int_0^\infty | 2\pi J_m(r) | \, | dv(r) |. \tag{8}$$

(7) and (8) show that

$$\| g^m \|_{A(S^1)} \geqq (\sup_{x \in \mathbb{R}} | J_m(x) |)^{-1}.$$

Here equality is attained as is shown by forming the Fourier-Stieltjes transform of a measure μ_m of the form (6) with v chosen as a suitable point measure at the point $r = r_0$, where $| J_m(r) |$ attains its maximal value.

Remark. Theorem 1 can be used to give information on the properties of pseudomeasures supported by curves Γ satisfying the condition of the theorem. A pseudomeasure is by definition an element of $PM(\mathbb{R}^n) = \mathscr{F}L^\infty(\mathbb{R}^n)$ and hence it is a distribution in $\mathscr{D}'(\mathbb{R}^n)$. If $\mu \in PM(\mathbb{R}^n)$ has its support included in Γ, we can also interpret μ as an element v of the dual of $C^1(\Gamma)$ in the sense that

$$\langle v, h \rangle = \langle \mu, g \rangle$$

for every $h \in C^1(\Gamma)$ and every $g \in C^1(\Gamma) \cap A(\mathbb{R}^2)$ with h as restriction to Γ. We refer to [3] for a brief motivation when $n = 2$; the case $n > 2$ is quite analogous.

The function f in Theorem 1 maps Γ into $\mathbb{R}$, $\phi \to \phi \circ f$ maps $C^1(\mathbb{R})$ into $C^1(\Gamma)$ and the adjoint mapping maps v into a distribution σ on $\mathbb{R}$, contained in the dual of $C^1(\mathbb{R})$. Now

$$\langle v, e^{itf} \rangle = \langle \sigma, \chi_t \rangle, \qquad t \in \mathbb{R},$$

where $\chi_t(x) = e^{itx}$, $x \in \mathbb{R}$, and since the extrapolations g constructed in the proof of Theorem 1 belong to C^1, we find that

$$| \langle \sigma, \chi_t \rangle | = | \langle \mu, g \rangle | \leqq \| \mu \|_{PM(\mathbb{R}^n)} \cdot K(1 + | t |)^{1/k}, \qquad t \in \mathbb{R}, \tag{9}$$

which gives an estimate of the Fourier transform of σ. It follows from (1) that the corresponding result is true with $k = 2$ for mappings of one-dimensional pseudomeasures, but (2) and the closed graph theorem show that in this case the exponent cannot be lowered in general. Hence (9) shows that pseudomeasures on curves of this type exhibit a kind of stability for highly differentiable mappings, which is not usually found in the space of one-dimensional pseudomeasures.

REFERENCES

1. A. Beurling, *Sur les intégrales de Fourier absolumentes convergentes et leur application à une transformation fonctionelle*, Neuvième congrès des mathématiciens scandinaves, Helsinki, 1938.

2. F. Carlson, *Une inégalité*, Ark. Mat. Astronom. Fys., 25, B1, 1934.

3. Y. Domar, Sur la synthèse harmonique des courbes de $\mathbb{R}^2$, C. R. Acad. Sci. Paris **270** (1970), 875–878.

4. Z. L. Leibenson, *On the ring of functions with absolutely convergent Fourier series*, Uspehi Matem. Nauk. (N.S.) 9 (61) (1954), 157–162.

5. H. S. Shapiro, *Topics in approximation theory*, Springer Lecture Notes 187, 1971.

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